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On the fuzzy Poisson equation

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Abstract

The fuzzy-valued vector function is defined and then divergence, Laplace, and gradient operators are defined for fuzzy-valued vector functions and fuzzy-valued functions. Moreover, a fuzzy Riemann line integral and its fundamental theorem are introduced. To complete our discussion, fuzzy Green's, fuzzy divergence, and fuzzy Green's identity theorems are proved. In detail, a fuzzy Poisson equation is considered by discussion of fuzzy maximum and minimum principles. Also, the uniqueness and stability of the solution of a fuzzy Poisson equation are investigated as theorems. Finally, for more illustration, some examples are solved.

Keywords: Fuzzy partial differential equation; Fuzzy Poisson equation; Fuzzy-valued vector function; Fuzzy Green's theorem; Fuzzy divergence theorem; Fuzzy Green's identity

1. Introduction

Differential equations with partial derivatives play an important role in modeling many physical phenomena, and in most branches of engineering. On the other hand, the lack of adequate information, uncertainties in the data, measurement errors, etc., result in uncertainties in provided models. These uncertainties in a differential equation can appear in every part of the equation, including the initial and boundary conditions, the coefficients in the equation, and the shape and dimension of the domain. Fuzzy modeling is an effective method for modeling problems with uncertainties in a way that can provide researchers with a more realistic view of the problem.

The fuzzy set theory was first introduced by Zadeh [25]. The concept of the fuzzy derivative was then studied by Chang and Zadeh [20]. Next, Dubois and Prade [10] extended the definition of the fuzzy derivative using Zadeh's extension principle. One of the first definitions of difference and derivative for set-valued functions (H-difference and H-derivative) was given by Hukuhara [13]. These concepts were then extended by many authors and researchers, such as Puri and Ralescu [19] and Kaleva [15] in fuzzy differential equations. Bede and Gal [4] framed the concepts of weak and strong generalized differentiability. Despite some advantages, these definitions were associated with a number of shortcomings. To alleviate this weakness, Stefanini [22] proposed a generalization of the Hukuhara difference. Then Stefanini and Bede [24] introduced generalized Hukuhara-type differentiability concepts of the interval-valued

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functions and studied interval differential equations with the differentiability concepts. Stefanini [23] developed a definition of difference called *gH-difference*, which was an extended definition of Hukuhara's difference. Later, Bede and Stefanini [5] suggested a newer definition. This difference, called *g-difference*, was proposed for fuzzy-valued functions.

The methods of solving fuzzy ordinary and fuzzy partial differential equations were extended, parallel to the development of fuzzy derivative definitions and concepts. From a historical point of view, fuzzy partial differential equations were first proposed by Buckley and Feuring [7]. Allahviranloo [1] proposed difference methods for solving partial differential equations, Faybishenko [11] proposed a fuzzy partial differential equation model for simulation of hydrogeologic system behavior, and Oberguggenberger [17] discussed weak and fuzzy solutions for the Burgers equation. In addition, Chen et al. [8] proposed an adaptive fuzzy algorithm for the calculation of fuzzy solutions of fuzzy partial differential equations. Furthermore, Jafelice et al. [14] introduced a fuzzy partial differential equation model for a biological phenomenon and computed its fuzzy solutions. The heat, wave, and Poisson equations, including fuzzy number parameters, were introduced in [6]. Allahviranloo et al. [2] considered a fuzzy heat equation and presented its fuzzy solution based on the generalized Hukuhaha differentiability.

The present article investigates the existence, uniqueness, and stability of a fuzzy Poisson equation with Dirichlet boundary conditions, and provides its fuzzy fundamental solution. This solution is obtained by use of a Green's function and the fuzzy Green's identity. Both the Poisson equation and the Laplace equation (in the special case) are second-order elliptic equations, and they are the reduced form of the Navier–Stokes equation, and are capable of modeling many problems in the field of engineering and fluid mechanics, such as groundwater flow with recharge or depletion and petroleum simulation and the mechanics of stretched, loaded membranes, and in the study of the theory of torsion of prismatic elastic bodies [21]. These reduced equations are used to give better insight to solve the more complicated Navier–Stokes equation with various boundary conditions for various solution domains. Because of the importance of such equations and according to the aforementioned explanations, investigating a fuzzy Poisson equation can provide us with a more comprehensive and precise view for solving several problems.

This article is organized as follows: First, Section 2 expresses some basic concepts of the fuzzy differentiability and the fuzzy integration and their properties. Furthermore, we prove some new properties to be used in the article. In Section 3 we introduce a fuzzy-valued vector function, fuzzy gradient, fuzzy divergence, and fuzzy Laplace operators, and their properties. Then we give some examples to illustrate their concepts. In Section 4 we propose a fuzzy line integral and its fundamental theorem. Then the fuzzy Green's theorem, fuzzy divergence theorem, and fuzzy Green's identity are proved. We then present some numerical examples to demonstrate their applications. Finally, in Section 5, a fuzzy Poisson equation with Dirichlet boundary conditions is studied, and then its uniqueness and stability are proved by use of the maximum principle. The fundamental fuzzy solution for a fuzzy Poisson equation is introduced, and then the fundamental solution for a fuzzy Poisson equation on a two-dimensional disc is studied in detail.

2. Preliminaries

In this section we review the basic definitions and the theorems that are used in this study. Furthermore, we prove some new concepts.

We denote $\mathbb{R}_{\mathcal{F}}$, the set of fuzzy numbers, that is, normal, fuzzy convex, upper semicontinuous, and compactly supported fuzzy sets that are defined over the real line. Let $u \in \mathbb{R}_{\mathcal{F}}$ be a fuzzy number; for $0 < \alpha \leq 1$, the α-level set (or α-cut) of u is defined by $[u]_\alpha = \left\{x \in \mathbb{R}^n \middle| u(x) \geq \alpha\right\}$, and for $\alpha = 0$ is defined by the closure of the support $[u]_0 = cl\left\{x \in \mathbb{R}^n \middle| u(x) > 0\right\}$. We denote $[u]_\alpha = [u_\alpha^-,\ u_\alpha^+]$, so the α-level set $[u]_\alpha$ is a closed interval for all $\alpha \in [0,\ 1]$.

If $u,\ v \in \mathbb{R}_{\mathcal{F}}$ and $\lambda \in \mathbb{R}$, the addition and the scalar multiplication are defined as having the α-levels of $[u + v]_\alpha = [u]_\alpha + [v]_\alpha$ and $[\lambda u]_\alpha = \lambda[u]_\alpha$, respectively.

A trapezoidal fuzzy number, denoted by $u = (a, b, c, d)$, where $a \leq b \leq c \leq d$, has α-cuts $[u]_\alpha = [a + \alpha(b - a), d - \alpha(d - c)]$ for $0 \leq \alpha \leq 1$; If $b = c$ we have a triangular fuzzy number. The support of fuzzy number u is defined as follows:

$$supp(u) = cl\left\{x \in \mathbb{R}^n \middle| u(x) > 0\right\},$$

where cl is the closure of set $\left\{x \in \mathbb{R}^n \middle| u(x) > 0\right\}$.

Definition 2.1. ([23]) Let $u,\ v \in \mathbb{R}_{\mathcal{F}}$ be two fuzzy numbers. Then the gH-difference is the fuzzy number k, if it exists, such that

$$u \ominus_{gH} v = k \Longleftrightarrow \begin{cases} \quad (i)\ u = v + k, \\ or\ (ii)\ v = u + (-1)k. \end{cases}$$

In terms of α-levels we have we have $[u \ominus_{gH} v]_\alpha = [\min\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\},\ \max\{u^-_\alpha - v^-_\alpha, u^+_\alpha - v^+_\alpha\}]$, and if the H-difference exists, then $u \ominus v = u \ominus_{gH} v$. The conditions for the existence of $w = u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$ are

$$case\ (i) \begin{cases} k^-_\alpha = u^-_\alpha - v^-_\alpha \text{ and } k^+_\alpha = u^+_\alpha - v^+_\alpha, & \forall\alpha \in [0,1], \\ \text{with } k^-_\alpha \text{ increasing, } k^+_\alpha \text{ decreasing, } k^-_\alpha \le k^+_\alpha, \end{cases}$$

$$case\ (ii) \begin{cases} k^-_\alpha = u^+_\alpha - v^+_\alpha \text{ and } k^+_\alpha = u^-_\alpha - v^-_\alpha, & \forall\alpha \in [0,1]. \\ \text{with } k^-_\alpha \text{ increasing, } k^+_\alpha \text{ decreasing, } k^-_\alpha \le k^+_\alpha. \end{cases}$$

It is easy to show that (i) and (ii) are both valid if and only if k is a crisp number.

Proposition 2.1. *([23]) Suppose that $u,\ v \in \mathbb{R}_{\mathcal{F}}$ are two fuzzy numbers. Then*

1. *If the gH-difference exists, it is unique.*
2. *$u \ominus_{gH} v = u \ominus v$ or $u \ominus_{gH} v = -(v \ominus u)$ whenever the statement on the right exists, especially, $u \ominus_{gH} u = u \ominus u = 0$.*
3. *If $u \ominus_{gH} v$ exists in sense (i), then $v \ominus_{gH} u$ exists in sense (ii) and vice versa.*
4. *$(u + v) \ominus_{gH} v = u$.*
5. *$0 \ominus_{gH} (u \ominus_{gH} v) = v \ominus_{gH} u$.*
6. *$u \ominus_{gH} v = v \ominus_{gH} u = k$ if and only if $k = -k$; moreover, $k = 0$ if and only if $u = v$.*

Definition 2.2. ([5]) The generalized difference (or g-difference for short) of two fuzzy numbers $u,\ v \in \mathbb{R}_{\mathcal{F}}$ is defined by its level sets as

$$[u \ominus_g v]_\alpha = cl \bigcup_{\beta \ge \alpha} ([u]_\beta \ominus_{gH} [v]_\beta), \qquad \forall\alpha \in [0,\ 1],$$

where the gH-difference $\ominus_{gH}$ is with interval operands $[u]_\beta$ and $[v]_\beta$.

Remark 1. We assume that $u \ominus_{gH} v \in \mathbb{R}_{\mathcal{F}}$ and furthermore $u \ominus_{gH} v = u \ominus_g v$.

Definition 2.3. ([16]) The Hausdorff distance between fuzzy numbers is given by $D : \mathbb{R}_{\mathcal{F}} \times \mathbb{R}_{\mathcal{F}} \longrightarrow \mathbb{R}^+ \cup \{0\}$ as

$$D(u,\ v) = \sup_{\alpha\in[0,\ 1]} d\Big([u]^\alpha\ ,\ [v]^\alpha\Big) = \sup_{\alpha\in[0,\ 1]} \max\Big\{|u^-(\alpha) - v^-(\alpha)|,\ |u^+(\alpha) - v^+(\alpha)|\Big\},$$

where d is the Hausdorff metric. The metric space $(\mathbb{R}_{\mathcal{F}},\ D)$ is complete, separable, and locally compact. Moreover, the Hausdorff distance has the following properties:

1. $D(u \oplus w,\ v \oplus w) = D(u,\ v),\ \forall u,\ v,\ w \in \mathbb{R}_{\mathcal{F}}$.
2. $D(\lambda u,\ \lambda v) = |\lambda| D(u,\ v),\ \forall\lambda \in \mathbb{R},\ u,\ v \in \mathbb{R}_{\mathcal{F}}$.
3. $D(u \oplus v,\ w \oplus z) \le D(u,\ w) + D(v,\ z),\ \forall u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.
4. $D(u \ominus v,\ w \ominus z) \le D(u,\ w) + D(v,\ z)$, as long as $u \ominus v$ and $w \ominus z$ exist, where $u,\ v,\ w,\ z \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.4. ([5]). Let $f : [a,b] \to \mathbb{R}_{\mathcal{F}}$ and $x_0 \in (a,b)$, with $f^-(x;\alpha)$ and $f^+(x;\alpha)$ both differentiable at x_0. Hence we say that

- f is $[i - gH]$-differentiable at x_0 if

$$f'_{i-gH}(x_0;\alpha) = [(f^-)'(x_0;\alpha)\ ,\ (f^+)'(x_0;\alpha)], \quad 0 \le \alpha \le 1, \tag{1}$$

- f is $[ii-gH]$-differentiable at x_0 if

$$f'_{ii-gH}(x_0;\alpha)=[(f^{+})'(x_0;\alpha)\ \ ,\ \ (f^{-})'(x_0;\alpha)],\qquad 0\le\alpha\le 1. \tag{2}$$

Definition 2.5. ([24]) We say that a point $x_0\in(a,b)$ is a switching point for the differentiability of f if in any neighborhood V of x_0 there exist points $x_1<x_0<x_2$ such that

type I at x_1 (1) holds while (2) does not hold and at x_2 (2) holds and (1) does not hold, or
type II at x_1 (2) holds while (1) does not hold and at x_2 (1) holds and (2) does not hold.

Definition 2.6. ([2].) Let $f:[a,b]\to\mathbb{R}_{\mathcal{F}}$ and $f'_{gH}(x)$ be gH-differentiable at $x_0\in(a,b)$. Moreover, there is no switching point on (a,b), and $(f^{-})'(x;\alpha)$ and $(f^{+})'(x;\alpha)$ are both differentiable at x_0. We say that

- $f'_{gH}(x)$ is $[i\text{-}gH]$-differentiable whenever the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ is the same:

$$f''_{i-gH}(x_0;\alpha)=[(f^{-})''(x_0;\alpha)\ \ ,\ \ (f^{+})''(x_0;\alpha)],\qquad 0\le\alpha\le 1;$$

- $f'_{gH}(x)$ is $[ii-gH]$-differentiable if the type of gH-differentiability $f(x)$ and $f'_{gH}(x)$ is different:

$$f''_{ii-gH}(x_0;\alpha)=[(f^{+})''(x_0;\alpha)\ \ ,\ \ (f^{-})''(x_0;\alpha)],\qquad 0\le\alpha\le 1.$$

Definition 2.7. ([9].) Let $f:[a,\ b]\to\mathbb{R}_{\mathcal{F}}$. We say that $f(x)$ is fuzzy Riemann integrable in $\mathbb{I}\in\mathbb{R}_F$ if for any $\varepsilon>0$ there exists $\delta>0$ such that for any division $P=\{[u,\ v];\xi\}$ with the norms $\Delta(P)<\delta$ we have

$$D\left(\sum_{p}^{*}(v-u)\odot f(\xi),\ \mathbb{I}\right)<\varepsilon,$$

where $\sum_p^*$ denotes the fuzzy summation and $\mathbb{I}$ indicates $\int_a^b f(x)dx$.

Theorem 2.2. *([5].) If f is gH-differentiable with no switching point in the interval $[a,b]$, then we have*

$$\int_a^b f'_{gH}(x)dx=f(b)\ominus_{gH} f(a).$$

Theorem 2.3. *([12]) Let $I\subseteq\mathbb{R}$ be an open interval and $x\in I$. Let $f:I\to\mathbb{R}_{\mathcal{F}}$ and $g:I\to\mathbb{R}$. Suppose that $g(x)$ is differentiable at x and the fuzzy function $f(x)$ is gH-differentiable at x. Then*

$$(f\odot g)'_{gH}(x)=f'_{gH}(x)\odot g(x)\oplus f(x)\odot g'(x). \tag{3}$$

Theorem 2.4. *([2].) Let I be an open interval in $\mathbb{R}$. Consider $g:I\to\zeta:=g(I)\subseteq\mathbb{R}$ is differentiable at x, and $f:\zeta\to\mathbb{R}_{\mathcal{F}}$ is gH-differentiable at the point $g(x)$. Then we have following conditions:*
If $g'(x)>0$

$$(f\circ g)'_{i-gH}(x)=g'(x)\odot f'_{i-gH}(g(x)),$$
$$(f\circ g)'_{ii-gH}(x)=g'(x)\odot f'_{ii-gH}(g(x)).$$

If $g'(x)<0$

$$(f\circ g)'_{i-gH}(x)=g'(x)\odot f'_{ii-gH}(g(x)),$$
$$(f\circ g)'_{ii-gH}(x)=g'(x)\odot f'_{i-gH}(g(x)).$$

Definition 2.8. ([2].) A fuzzy-valued function f of two variables is a rule that assigns to each ordered pair of real numbers, (x, y), in a set $\mathbb{D}$ a unique fuzzy number denoted by $f(x, y)$. The set $\mathbb{D}$ is the domain of f and its range is the set of values that f takes on, that is, $\{f(x, y)|(x, y) \in \mathbb{D}\}$.

We show the parametric representation of the fuzzy-valued function $f : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ by $f(x, y; \alpha) = [f^-(x, y; \alpha), f^+(x, y; \alpha)]$, for all $(x, y) \in \mathbb{D}$ and $\alpha \in [0, 1]$.

Definition 2.9. ([2].) A fuzzy-valued function $f : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ is said to be fuzzy continuous at $(x_0, y_0) \in \mathbb{D}$ if $\lim_{(x,y)\to(x_0,y_0)} f(x, t) = f(x_0, y_0)$. We say f is fuzzy continuous on $\mathbb{D}$ if f is fuzzy continuous at every point (x_0, y_0) in $\mathbb{D}$.

Definition 2.10. ([2].) Let $(x_0, y_0) \in \mathbb{D}$. Then the first generalized Hukuhara partial derivatives ([gH-p]-derivatives for short) of a fuzzy-valued function $f(x, y) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ at (x_0, t_0) with respect to variables x, y are the functions $f_{x_{gH}}(x_0, y_0)$ and $f_{y_{gH}}(x_0, y_0)$ given by

$$f_{x_{gH}}(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)}{h},$$
$$f_{y_{gH}}(x_0, y_0) = \lim_{k\to 0} \frac{f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)}{k},$$

provided that $f_{x_{gH}}(x_0, y_0)$ and $f_{y_{gH}}(x_0, y_0) \in \mathbb{R}_{\mathcal{F}}$.

Definition 2.11. ([2].) A fuzzy-valued function $f(x, y) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$ at (x_0, y_0) with respect to variables x, y is [gH-p]-differentiable if

$$\lim_{h\to 0} D\Big(\frac{f(x_0 + h, y_0) \ominus_{gH} f(x_0, y_0)}{h}, f_{x_{gH}}(x_0, y_0)\Big) \to 0, \tag{4}$$
$$\lim_{k\to 0} D\Big(\frac{f(x_0, y_0 + k) \ominus_{gH} f(x_0, y_0)}{k}, f_{y_{gH}}(x_0, y_0)\Big) \to 0, \tag{5}$$

respectively.

Definition 2.12. ([2].) Let $f(x, y) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, and let $(x_0, y_0) \in \mathbb{D}$ and $f^-(x, y; \alpha)$ and $f^+(x, y; \alpha)$ be real-valued functions that are partially differentiable with respect to. x. We say that

- $f(x, y)$ is $[i - p]$-differentiable with respect to x at (x_0, y_0) if
$$f_{x_{i-gH}}(x_0, y_0; \alpha) = [f_x^-(x_0, y_0; \alpha), \ f_x^+(x_0, y_0; \alpha)]. \tag{6}$$
- $f(x, y)$ is $[ii - p]$-differentiable with respect to x at (x_0, y_0) if
$$f_{x_{ii-gH}}(x_0, y_0; \alpha) = [f_x^+(x_0, y_0; \alpha), \ f_x^-(x_0, y_0; \alpha)]. \tag{7}$$

Definition 2.13. ([2].) For any fixed ξ_0, we say that $(\xi_0, y) \in \mathbb{D}$ is a switching point for the differentiability of $f(x, y)$ with respect to x if in any neighborhood V of (ξ_0, y) there exist points $(x_1, y) < (\xi_0, y) < (x_2, y)$ such that

Type I at (x_1, y) (6) holds while (7) does not hold and at (x_2, y) (7) holds and (6) does not hold for all y, or
Type II at (x_1, y) (7) holds while (6) does not hold and at (x_2, y) (6) holds and (7) does not hold for all y.

Note 1. In this article we assume that we do not have a switching point unless mentioned otherwise.

Definition 2.14. ([2].) Let $f(x, y) : \mathbb{D} \to \mathbb{R}_{\mathcal{F}}$, and let $\partial_x f(x, y)$ be [gH-p]-differentiable at $(x_0, y_0) \in \mathbb{D}$ with respect to x. Moreover, there is no switching point on $\mathbb{D}$. We say that

- $f_{x_{gH}}(x, y)$ is $[i - p]$-differentiable with respect to x if the type of [gH-p]-differentiability of both $f(x, y)$ and $f_{x_{gH}}(x, y)$ is the same:
$$f_{xx_{i-gH}}(x_0, y_0; \alpha) = [f_{xx}^-(x_0, y_0; \alpha), \ f_{xx}^+(x_0, y_0; \alpha)].$$

- $f_{x_{gH}}(x,y)$ is $[ii-p]$-differentiable with respect to x if the type of [gH-p]-differentiability $f(x,y)$ and $f_{x_{gH}}(x,y)$ is different:

$$f_{xx_{ii-gH}}(x_0,y_0;\alpha)=[f^{+}_{xx}(x_0,y_0;\alpha),\ f^{-}_{xx}(x_0,y_0;\alpha)].$$

Lemma 2.1. *([2].) Consider $f:\mathbb{D}\to\mathbb{R}_{\mathcal{F}}$ as a fuzzy continuous function. Assume that f is [gH-p]-differentiable with respect to x, with no switching point in the interval $[a,b]$, and fuzzy continuous, then we have*

$$\int_a^b f_{y_{gH}}(x,y)\,dy=f(x,b)\ominus_{gH}f(x,a).$$

Theorem 2.5. *([2].) Let f be a fuzzy-valued function such that $f'_{gH}(c)=0$ and the second gH-derivative of f exists on an open interval containing c.*

1. *If $f''_{gH}(c)\succ 0$, then c is a local minimum of f.*
2. *If $f''_{gH}(c)\prec 0$, then c is a local maximum of f.*
3. *If $0\in supp\ (f''_{gH}(c))$, the test is inconclusive.*

Lemma 2.2. *$\int_b^a f(x,y)\,dx=\ominus\int_a^b f(x,y)\,dx$, where $\ominus$ denotes the Hukuhara difference and $f(x,y)$ is a fuzzy-valued function.*

Proof. According to Lemma 2.1 we know that

$$\int_a^b f(x,y)\,dx=F(b,y)\ominus_{gH}F(a,y),$$

where $F_{x_{gH}}(x,y)=f(x,y)$. Without loss of generality, suppose that $\ominus_{gH}$ is case (i), so in terms of α-levels we have

$$\left[\int_a^b f(x,y)\,dx\right]_\alpha=\left[\int_a^b f^-(x,y;\alpha)dx\ ,\ \int_a^b f^+(x,y;\alpha)dx\right]$$
$$=[F^-(b,y;\alpha)-F^-(a,y;\alpha)\ ,\ F^+(b,y;\alpha)-F^+(a,y;\alpha)],$$

taking into consideration $0\ominus\int_a^b f(x,y)\,dx=\ominus\int_a^b f(x,y)\,dx$, where 0 is a singleton. Therefore in terms of α-levels we have

$$\left[-\int_a^b f^-(x,y;\alpha)dx\ ,\ -\int_a^b f^+(x,y;\alpha)dx\right]=[F^-(a,y;\alpha)-F^-(b,y;\alpha)\ ,\ F^+(a,y;\alpha)-F^+(b,y;\alpha)]$$
$$=\left[\int_b^a f^-(x,y;\alpha)dx\ ,\ \int_b^a f^+(x,y;\alpha)dx\right]=\left[\int_b^a f(x,y)\,dx\right]_\alpha.$$

The proof of case (ii) is similar. □

Corollary 1. *Under the assumption of Lemma 2.2, we also have $\int_b^a f(x,y)\,dy=\ominus\int_a^b f(x,y)\,dy$.*

Definition 2.15. Let $Z:=f(x,y)$ be a fuzzy-valued function. Then f is gH-differentiable at (x_0,y_0) if $\triangle Z$ can be expressed in the form

$$\triangle Z=f_{x_{gH}}(x_0,y_0)\odot\triangle x\oplus f_{y_{gH}}(x_0,y_0)\odot\triangle y+\varepsilon_1\triangle x+\varepsilon_2\triangle y, \tag{8}$$

where $\varepsilon_1\to 0$ and $\varepsilon_2\to 0$ as $(\triangle x,\triangle y)\to 0$.

Note 2. Let $\triangle Z = f(x + \triangle x,\ y + \triangle y) \ominus_{gH} f(x, y)$ and $\triangle x = x(t + \triangle t) - x(t)$ and $\triangle y = y(t + \triangle t) - y(t)$.

Theorem 2.6. *(The chain rule) Let $Z := f(x, y)$ be a fuzzy-valued function, where $x = x(t)$ and $y = y(t)$ are differentiable real-valued functions of t. Then f is a gH-differentiable function of t and we have*

$$\frac{dz}{dt} = f_{x_{gH}} \odot \frac{dx}{dt} \oplus f_{y_{gH}} \odot \frac{dy}{dt}. \tag{9}$$

Proof. By dividing both sides of equation (8) by $\triangle t$, we obtain

$$\frac{\triangle Z}{\triangle t} = f_{x_{gH}} \odot \frac{\triangle x}{\triangle t} \oplus f_{y_{gH}} \odot \frac{\triangle x}{\triangle t} + \varepsilon_1 \frac{\triangle x}{\triangle t} + \varepsilon_2 \frac{\triangle y}{\triangle t}.$$

Let $\triangle t \to 0$; then $\triangle x, \triangle y \to 0$ because of differentiability of $x(t)$ and $y(t)$. This means that $\varepsilon_1, \varepsilon_2 \to 0$, and therefore the following result is obtained:

$$\begin{aligned}
\frac{dZ}{dt} &= \lim_{\triangle t \to 0} \frac{\triangle Z}{\triangle t} \\
&= f_{x_{gH}} \odot \lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t} \oplus f_{y_{gH}} \odot \lim_{\triangle t \to 0} \frac{\triangle y}{\triangle t} + \lim_{\triangle t \to 0} \varepsilon_1 \lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t} + \lim_{\triangle t \to 0} \varepsilon_1 \lim_{\triangle t \to 0} \frac{\triangle x}{\triangle t} \\
&= f_{x_{gH}} \odot \frac{dx}{dt} \oplus f_{y_{gH}} \odot \frac{dy}{dt} + 0 \cdot \frac{dx}{dt} + 0 \cdot \frac{dy}{dt} \\
&= f_{x_{gH}} \odot \frac{dx}{dt} \oplus f_{y_{gH}} \odot \frac{dy}{dt}. \quad \square
\end{aligned}$$

Remark 2. If $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j},\ a \le t \le b$, is a parametric curve where $x(t)$ and $y(t)$ are the real-valued functions and $f(x, y)$ is a fuzzy-valued function, by setting $z := f(r(t))$, and according to Theorem 2.6, we obtain

$$\frac{f(r(t))}{dt} = f_{x_{gH}}(x, y) \odot \frac{dx}{dt} \oplus f_{y_{gH}}(x, y) \odot \frac{dy}{dt}.$$

Lemma 2.3. *Assume that $\int_a^b f(x, y)\, dx$ and $\int_a^b g(x, y)\, dx$ exist, where $f(x, y)$ and $g(x, y)$ are the fuzzy-valued functions and $H(x, y) \in \mathbb{R}_{\mathcal{F}}$ is defined as follows:*

$$H(x, y) = \begin{cases} f(x, y) & a \le b, \\ g(x, y) & b \le c. \end{cases}$$

Then we have

$$\int_a^c H(x, y)\, dx = \int_a^b f(x, y)\, dx \oplus \int_b^c g(x, y)\, dx.$$

Proof. By use of the fuzzy integral properties and the definition of $H(x, y)$, it is obvious that

$$\begin{aligned}
\int_a^c H(x, y)\, dx &= \int_a^b H(x, y)\, dx \oplus \int_b^c H(x, y)\, dx \\
&= \int_a^b f(x, y)\, dx \oplus \int_b^c g(x, y)\, dx.
\end{aligned}$$

See [3]. $\square$

Remark 3. These properties can be generalized for finite number integrals.

Lemma 2.4. *If $f(x, y)$ and $g(x, y)$ are the fuzzy-valued functions, then we have*

$$\int f(x, y)\, dx \ominus_{gH} \int g(x, y)\, dx = \int \Big(f(x, y) \ominus_{gH} g(x, y) \Big)\, dx. \tag{10}$$

Proof. First we set $F := \int f(x, y)\, dx$ and $G := \int g(x, y)\, dx$ for short, and then according to Definition 2.1

$$F \ominus_{gH} G = K \Rightarrow \begin{cases} (i) & F = G \oplus K, \\ (ii) & G = F \oplus (-1)k. \end{cases}$$

In terms of α-levels we have the following statements:
In case (i): $[F_\alpha^-, F_\alpha^+] = [G_\alpha^- + K_\alpha^-, G_\alpha^+ + K_\alpha^+]$, so $K_\alpha^- = F_\alpha^- - G_\alpha^-$ and $K_\alpha^+ = F_\alpha^+ - G_\alpha^+$.
In case (ii): $[G_\alpha^-, G_\alpha^+] = [F_\alpha^- - K_\alpha^+, F_\alpha^+ - K_\alpha^-]$, so $K_\alpha^- = F_\alpha^+ - G_\alpha^+$ and $K_\alpha^+ = F_\alpha^- - G_\alpha^-$.
If on the right side of equation (10), we put $f := f(x, y)$ and $g := g(x, y)$, by Definition 2.1 we have

$$f \ominus_{gH} g = k \Rightarrow \begin{cases} (i) & f = g \oplus k, \\ (ii) & g = f \oplus (-1)k. \end{cases}$$

In terms of α-levels we have the following statements:
In case (i): $k_\alpha^- = f_\alpha^- - g_\alpha^-$ and $k_\alpha^+ = f_\alpha^+ - g_\alpha^+$. So

$$\begin{aligned}
\left[\int k\, dx\right]_\alpha &= \left[\int k_\alpha^- dx\, ,\ \int k_\alpha^+ dx\right] \\
&= \left[\int \left(f_\alpha^- - g_\alpha^-\right) dx\, ,\ \int \left(f_\alpha^+ - g_\alpha^+\right) dx\right] \\
&= \left[\int f_\alpha^-\, dx - \int g_\alpha^-\, dx\, ,\ \int f_\alpha^+\, dx - \int g_\alpha^+\, dx\right] \\
&= [F_\alpha^- - G_\alpha^-\, ,\ F_\alpha^+ - G_\alpha^+] = [K_\alpha^-\, ,\ K_\alpha^+].
\end{aligned}$$

So (10) for case (i) exists.
In case (ii): $k_\alpha^- = f_\alpha^+ - g_\alpha^+$ and $k_\alpha^+ = f_\alpha^- - g_\alpha^-$. So

$$\begin{aligned}
\left[\int k\, dx\right]_\alpha &= \left[\int k_\alpha^- dx\, ,\ \int k_\alpha^+ dx\right] \\
&= \left[\int \left(f_\alpha^+ - g_\alpha^+\right) dx\, ,\ \int \left(f_\alpha^- - g_\alpha^-\right) dx\right] \\
&= \left[\int f_\alpha^+\, dx - \int g_\alpha^+\, dx\, ,\ \int f_\alpha^-\, dx - \int g_\alpha^-\, dx\right] \\
&= [F_\alpha^+ - G_\alpha^+\, ,\ F_\alpha^- - G_\alpha^-] = [K_\alpha^-\, ,\ K_\alpha^+].
\end{aligned}$$

So (10) for case (ii) exists. □

Lemma 2.5. *Suppose that if $u \in \mathbb{R}_\mathcal{F}$ and v is a singleton, then $u \oplus (-1)v = u \ominus_{gH} v$ in the sense of case (i), and if u is a singleton and $v \in \mathbb{R}_\mathcal{F}$, then $u \oplus (-1)v = u \ominus_{gH} v$ in the sense of case (ii).*

Proof. First, we suppose that case (i) holds. In terms of α-levels we have

$$\begin{aligned}
[u \oplus (-1)v]_\alpha &= [u_\alpha^- + (-1)v_\alpha^+\, ,\ u_\alpha^+ + (-1)v_\alpha^-] \\
&= [u_\alpha^- - v_\alpha^+\, ,\ u_\alpha^+ - v_\alpha^-].
\end{aligned}$$

Also in case (i) we have

$$[u \ominus_{gH} v]_\alpha = [u_\alpha^- - v_\alpha^-, u_\alpha^+ - v_\alpha^+].$$

Because v is a singleton, $v^- = v^+$. Hence the first statement holds. Now we assume that case (ii) exists. In terms of α-levels we get

$$[u \ominus_{gH} v]_\alpha = [u^+_\alpha - v^+_\alpha, u^-_\alpha - v^-_\alpha].$$

Because u is a singleton, $u^- = u^+$. Thus the second statement holds. □

3. Fuzzy-valued vector function

In what follows, we first define a fuzzy-valued vector function and introduce its properties. Then we propose a fuzzy gradient, a fuzzy divergence, and a fuzzy Laplace operator. We end this section with some examples to illustrate their concepts.

Definition 3.1. Let D be a set in $\mathbb{R}^2$ ($\mathbf{D} \subseteq \mathbb{R}^2$); $p(x,y)$ and $q(x,y)$: $\mathbf{D} \to \mathbb{R}_\mathcal{F}$ are the fuzzy-valued functions and $\mathbf{i} = \langle 1, 0\rangle$ and $\mathbf{j} = \langle 0, 1\rangle$ are standard basis vectors on a two-dimensional plane region. $F(x, y)$ is a two-dimensional fuzzy-valued vector function, and we can show it in terms of its fuzzy-valued function components as follows:

$$F(x,y) = p(x,y)\mathbf{i} \oplus q(x,y)\mathbf{j} = \langle p(x,y)\ ,\ q(x,y)\rangle$$

(or for short, $F = p\mathbf{i} \oplus q\mathbf{j} = \langle p,\ q\rangle$). This means that $p(x, y),\ q(x, y) \in \mathbb{R}_\mathcal{F}$ are fuzzy-valued function components on the x-coordinate and the y-coordinate, respectively. As a result, $F \in \mathbb{R}_\mathcal{F} \times \mathbb{R}_\mathcal{F}$.

Note 3. Briefly $\langle\ .\ ,\ .\ \rangle$ denotes the vector-valued function.

Proposition 3.1. *Let $F = \langle f_1,\ f_2\rangle$ and $G = \langle g_1,\ g_2\rangle$ be two fuzzy-valued vector functions under the assumption of Definition 3.1, and $c \in \mathbb{R}$ is a scalar. We have*

1. $F \oplus G = \langle f_1 \oplus g_1,\ f_2 \oplus g_2\rangle$.
2. $c \odot F = \langle c \odot f_1,\ c \odot f_2\rangle$.
3. $F \ominus_{gH} G = \langle f_1 \ominus_{gH} g_1,\ f_2 \ominus_{gH} g_2\rangle$.

Proof. In the sense of Definition 3.1 $f_1, f_2, g_1, g_2 : \mathbf{D} \to \mathbb{R}_\mathcal{F}$ and $f_1^-(x, y; \alpha)$, $f_1^+(x, y; \alpha)$, $f_2^-(x, y; \alpha)$, $f_2^+(x, y; \alpha)$, $g_1^-(x, y; \alpha)$, $g_1^+(x, y; \alpha)$, $g_2^-(x, y; \alpha)$, and $g_2^+(x, y; \alpha)$ are the real-valued functions for all $(x, y) \in \mathbf{D}$ and $\alpha \in (0, 1]$. Then

1. $F \oplus G = \langle f_1, f_2\rangle \oplus \langle g_1, g_2\rangle$. In terms of α-levels we have

$$\begin{aligned}
[F \oplus G]_\alpha &= F_\alpha + G_\alpha \\
&= \big\langle (f_1^-(x, y; \alpha), f_1^+(x, y; \alpha))\ ,\ (f_2^-(x, y; \alpha), f_2^+(x, y; \alpha))\big\rangle \\
&\quad + \big\langle (g_1^-(x, y; \alpha), g_1^+(x, y; \alpha))\ ,\ (g_2^-(x, y; \alpha), g_2^+(x, y; \alpha))\big\rangle \\
&= \big\langle (f_1^-(x, y; \alpha), f_1^+(x, y; \alpha)) + (g_1^-(x, y; \alpha), g_1^+(x, y; \alpha))\ , \\
&\quad (f_2^-(x, y; \alpha), f_2^+(x, y; \alpha)) + (g_2^-(x, y; \alpha), g_2^+(x, y; \alpha))\big\rangle \\
&= \big\langle (f_1^-(x, y; \alpha) + g_1^-(x, y; \alpha), f_1^+(x, y; \alpha) + g_1^+(x, y; \alpha))\ , \\
&\quad (f_2^-(x, y; \alpha) + g_2^-(x, y; \alpha), f_2^+(x, y; \alpha) + g_2^+(x, y; \alpha))\big\rangle \\
&= \big\langle f_1(x, y; \alpha) + g_1(x, y; \alpha)\ ,\ f_2(x, y; \alpha) + g_2(x, y; \alpha)\big\rangle\ .
\end{aligned}$$

Hence the first statement exists.

2. For second case, first we suppose that $c \geq 0$:

$$\begin{aligned}
[c \odot F]_\alpha &= c \cdot F_\alpha \\
&= c \cdot \big\langle f_1(x, y; \alpha)\ ,\ f_2(x, y; \alpha)\big\rangle \\
&= c \cdot \big\langle (f_1^-(x, y; \alpha), f_1^+(x, y; \alpha))\ ,\ (f_2^-(x, y; \alpha), f_2^+(x, y; \alpha))\big\rangle
\end{aligned}$$

$$
\begin{aligned}
&= \langle c \cdot (f_1^-(x,y;\alpha), f_1^+(x,y;\alpha)) \ , \ c \cdot (f_2^-(x,y;\alpha), f_2^+(x,y;\alpha)) \rangle \\
&= \langle (c \cdot f_1^-(x,y;\alpha), c \cdot f_1^+(x,y;\alpha)) \ , \ (c \cdot f_2^-(x,y;\alpha), c \cdot f_2^+(x,y;\alpha)) \rangle \\
&= \langle c \cdot f_1(x,y;\alpha) \ , \ c \cdot f_2(x,y;\alpha) \rangle \ .
\end{aligned}
$$

If we suppose that $c < 0$, then we have

$$
\begin{aligned}
[c \odot F]_\alpha &= c \cdot F_\alpha \\
&= c \cdot \langle f_1(x,y;\alpha) \ , \ f_2(x,y;\alpha) \rangle \\
&= c \cdot \langle (f_1^-(x,y;\alpha), f_1^+(x,y;\alpha)) \ , \ (f_2^-(x,y;\alpha), f_2^+(x,y;\alpha)) \rangle \\
&= \langle c \cdot (f_1^-(x,y;\alpha), f_1^+(x,y;\alpha)) \ , \ c \cdot (f_2^-(x,y;\alpha), f_2^+(x,y;\alpha)) \rangle \\
&= \langle (c \cdot f_1^+(x,y;\alpha), c \cdot f_1^-(x,y;\alpha)) \ , \ (c \cdot f_2^+(x,y;\alpha), c \cdot f_2^-(x,y;\alpha)) \rangle \\
&= \langle c \cdot f_1(x,y;\alpha) \ , \ c \cdot f_2(x,y;\alpha) \rangle \ .
\end{aligned}
$$

Therefore the second statement is true.

3. For last case, based on Definition 2.1 we have

$$
F \ominus_{gH} G = K \ \Rightarrow \begin{cases} (i) & F = K \oplus G, \\ (ii) & G = F \oplus (-1) \cdot K, \end{cases}
$$

where $K \in \mathbb{R}_{\mathcal{F}}$ is fuzzy-valued vector function and $K = \langle k_1, \ k_2 \rangle$. If case (i) exists, then

$$
F = K \oplus G = \langle k_1 \ , \ k_2 \rangle \oplus \langle g_1 \ , \ g_2 \rangle,
$$

and on account of the first part of Proposition 3.1 we obtain

$$
F = \langle f_1 \ , \ f_2 \rangle = \langle k_1 \oplus g_1 \ , \ k_2 \oplus g_2 \rangle \ .
$$

If case (ii) exists, we have

$$
G = F \oplus (-1) \cdot k = \langle f_1 \ , \ f_2 \rangle \oplus (-1) \cdot \langle k_1 \ , \ k_2 \rangle.
$$

In terms of α-levels we have $\big[\langle f_1, \ f_2 \rangle \oplus (-1) \cdot \langle k_1, \ k_2 \rangle\big]_\alpha$. By using the first and the second parts of Proposition 3.1, we obtain

$$
\begin{aligned}
&\langle \big(f_1^-(x,y;\alpha), f_1^+(x,y;\alpha)\big) \ , \ \big(f_2^-(x,y;\alpha), f_2^+(x,y;\alpha)\big) \rangle \\
&+ \langle \big(-k_1^+(x,y;\alpha) \ , \ -k_1^-(x,y;\alpha)\big) \ , \ \big(-k_2^+(x,y;\alpha) \ , \ -k_2^-(x,y;\alpha)\big) \rangle \\
= &\Big\langle \Big(f_1^-(x,y;\alpha) + \big(-k_1^+(x,y;\alpha)\big), f_1^+(x,y;\alpha) + \big(-k_1^-(x,y;\alpha)\big) \Big) \ , \\
&\Big(f_2^-(x,y;\alpha) + \big(-k_2^+(x,y;\alpha)\big), f_2^+(x,y;\alpha) + \big(-k_2^-(x,y;\alpha)\big) \Big) \Big\rangle \\
= &\langle f_1(x,y;\alpha) + (-1)k_1(x,y;\alpha) \ , \ f_2(x,y;\alpha) + (-1)k_2(x,y;\alpha) \rangle \\
= &\langle g_1(x,y;\alpha) \ , \ g_2(x,y;\alpha) \rangle \ .
\end{aligned}
$$

Hence the third part of this proposition is established. □

Definition 3.2. (The inner product between two fuzzy-valued vector functions) If $F(x,y) = \langle f_1, \ f_2 \rangle$ and $G(x,y) = \langle g_1, \ g_2 \rangle$ are two fuzzy-valued vector functions defined by Definition 3.1, then the inner product of F and G is a fuzzy-valued function, given by

$$
F \circledcirc G = (f_1 \odot g_1) \oplus (f_2 \odot g_2) \ ,
$$

where $\circledcirc$ is fuzzy inner product of F and G.

We define two fuzzy-valued functions $P(x,y)$ and $Q(x,y)$ by $f_1 \odot g_1 := P$ and $f_2 \odot g_2 := Q$. Then we observe that for all $0 < \alpha \leq 1$

$$[f_1 \odot g_1]_\alpha = [P]_\alpha = [P_\alpha^- \ , \ P_\alpha^+],$$

where

$$P_\alpha^- = \min\{u \cdot v \ | \ u \in [f_1]_\alpha \ , \ v \in [g_1]_\alpha\},$$
$$P_\alpha^+ = \max\{u \cdot v \ | \ u \in [f_1]_\alpha \ , \ v \in [g_1]_\alpha\}.$$

Furthermore,

$$[f_2 \odot g_2]_\alpha = [Q]_\alpha = [Q_\alpha^- \ , \ Q_\alpha^+],$$

where

$$Q_\alpha^- = \min\{u \cdot v \ | \ u \in [f_2]_\alpha \ , \ v \in [g_2]_\alpha\},$$
$$Q_\alpha^+ = \max\{u \cdot v \ | \ u \in [f_2]_\alpha \ , \ v \in [g_2]_\alpha\}.$$

Hence for all $0 < \alpha \leq 1$ we have

$$[F \circledcirc G]_\alpha = [P_\alpha^- \ , \ P_\alpha^+] + [Q_\alpha^- \ , \ Q_\alpha^+].$$

Definition 3.3. (The inner product between a real-valued vector function and a fuzzy-valued vector function) If $F(x, y) = \langle f_1, \ f_2 \rangle$ is a fuzzy-valued vector function defined by Definition 3.1 and $h(x, y)$ is a real-valued vector function given by $h(x, y) = h_1(x, y)\mathbf{i} + h_2(x, y)\mathbf{j}$, where $h_1, \ h_2 \in \mathbb{R}$, then the inner product between h and F is a fuzzy-valued function given by

$$h \circledcirc F = (h_1 \odot f_1) \oplus (h_2 \odot f_2).$$

In terms of α-levels $[F]_\alpha = \langle [f_1]_\alpha, \ [f_2]_\alpha \rangle$. We have

$$h_1 \cdot [f_1]_\alpha = \begin{cases} \left(h_1 \cdot f_{\alpha_1}^- \ , \ h_1 \cdot f_{\alpha_1}^+\right), & h_1(x, y) \geq 0, \\ \left(h_1 \cdot f_{\alpha_1}^+ \ , \ h_1 \cdot f_{\alpha_1}^-\right), & h_1(x, y) < 0 \end{cases}$$

and

$$h_2 \cdot [f_2]_\alpha = \begin{cases} \left(h_2 \cdot f_{\alpha_2}^- \ , \ h_2 \cdot f_{\alpha_2}^+\right), & h_2(x, y) \geq 0, \\ \left(h_2 \cdot f_{\alpha_2}^+ \ , \ h_2 \cdot f_{\alpha_2}^-)\right), & h_2(x, y) < 0. \end{cases}$$

As a consequence we obtain $h(x, y) \cdot [F]_\alpha = h_1 \cdot [f_1]_\alpha + h_2 \cdot [f_2]_\alpha$.

Definition 3.4. (Gradient of a fuzzy-valued function) If $f(x, y) \in \mathbb{R}_\mathcal{F}$ is a fuzzy-valued function on $\mathbb{R}^2$, the gradient of $f(x, y)$, or grad $f(x, y)$, is defined by

$$\text{grad}\, f(x, y) = \nabla f(x, y) = f_x(x, y)\mathbf{i} \oplus f_y(x, y)\mathbf{j} = \langle f_x \ , \ f_y \rangle.$$

This means that grad $f(x, y)$ is a fuzzy-valued vector function and we have the following conditions:

- If f_x and f_y are $[i - p]$-gH differentiable at (x_0, y_0), grad $f(x, y)$ is denoted by $\nabla_{1,1} f$.
- If f_x and f_y are $[ii - p]$-gH differentiable at (x_0, y_0), grad $f(x, y)$ is denoted by $\nabla_{2,2} f$.
- If f_x is $[i - p]$-gH differentiable and f_y is $[ii - p]$-gH differentiable at (x_0, y_0), grad $f(x, y)$ is denoted by $\nabla_{1,2} f$.
- If f_x is $[ii - p]$-gH differentiable and f_y is $[i - p]$-gH differentiable at (x_0, y_0), grad $f(x, y)$ is denoted by $\nabla_{2,1} f$.

Note 4. The gradient of the fuzzy-valued function is denoted briefly by grad $f(x, y) = \nabla_{i,j} f$, $\{i, j\} \in \{1, 2\}$.

Note 5. $\nabla = (\frac{\partial}{\partial x})\mathbf{i} \oplus (\frac{\partial}{\partial y})\mathbf{j}$ is a linear operator on the fuzzy-valued functions.

Remark 4. Since in some cases we have no idea about the type of gradient of fuzzy-valued functions, we generally use ∇f for short.

Definition 3.5. (Divergence of a fuzzy-valued vector function) If $F(x,y) = P(x,y)\mathbf{i} \oplus Q(x,y)\mathbf{j}$ is a fuzzy-valued vector function defined by Definition 3.1, divergence of $F(x,y)$, or div F, is defined by $\operatorname{div} F(x,y) = P_x(x,y) \oplus Q_x(x,y)$. In terms of the gradient operator defined by Remark 5, the divergence of $F(x,y)$ can be written symbolically as the fuzzy inner product of ∇ and $F(x,y)$ as follows:

$$\operatorname{div} F = \nabla \circledcirc F.$$

Note 6. $\operatorname{div} = (\frac{\partial}{\partial x}) \oplus (\frac{\partial}{\partial y})$ is an operator.

Since ∇f is a fuzzy-valued vector function, by Definition 3.3 we obtain the following properties:

$$\begin{aligned}\operatorname{div}(\nabla f) &= \nabla \circledcirc (\nabla f)\\ &= \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right) \oplus \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)\\ &= \frac{\partial^2 f}{\partial x^2} \oplus \frac{\partial^2 f}{\partial y^2}.\end{aligned}$$

We can express these properties as $\nabla^2 f$, where $\nabla^2 = \nabla \circledcirc \nabla$. ∇^2 is called the *Laplace operator*. We can apply ∇^2 to fuzzy vector $F(x,y) = P(x,y)\mathbf{i} \oplus Q(x,y)\mathbf{j}$ in terms of its components. Hence we have $\nabla^2 F = \nabla^2 P \oplus \nabla^2 Q$.

Lemma 3.1. *Let $U(x,y)$ be a fuzzy-valued function and $g(x,y)$ be a real-valued function, where $\frac{\partial g}{\partial x}$ and $\frac{\partial g}{\partial y}$ have the same sign. Then*

$$\nabla \circledcirc (U \odot g) = \nabla_{i,j} U \odot g \oplus U \odot \nabla g,$$

where $\{i, j\} \in \{1, 2\}$.

Proof. According to Definition 3.5 and Theorem 2.3 and because of the linearity of the divergence operator we have

$$\begin{aligned}(\frac{\partial}{\partial x} \oplus \frac{\partial}{\partial y}) \circledcirc (U \odot g) &= \frac{\partial}{\partial x}(U \odot g) \oplus \frac{\partial}{\partial y}(U \odot g)\\ &= \left((\frac{\partial U}{\partial x} \odot g) \oplus (U \odot \frac{\partial g}{\partial x})\right) \oplus \left((\frac{\partial U}{\partial y} \odot g) \oplus (U \odot \frac{\partial g}{\partial y})\right)\\ &= \left((\frac{\partial}{\partial x} \oplus \frac{\partial}{\partial y}) \circledcirc U\right) \odot g \oplus \left(U \odot (\frac{\partial g}{\partial x} + \frac{\partial g}{\partial y})\right)\\ &= \nabla_{i,j} U \odot g \oplus U \odot \nabla g,\end{aligned}$$

for $\{i, j\} \in \{1, 2\}$. □

Definition 3.6. (Laplace operator of a fuzzy-valued function) If $f(x,y)$ is a fuzzy-valued function, and if $f(x,y)$ is $[i, ii-p]$-gH differentiable and f_{xx} and f_{yy} exist, $\nabla^2 f(x,y) = \frac{\partial^2 f}{\partial x^2} \oplus \frac{\partial^2 f}{\partial y^2}$ is the Laplace operator of $f(x,y)$. Therefore we have four conditions:

- If f_{xx} and f_{yy} are $[i-p]$-gH differentiable, the Laplace operator of $f(x,y)$ is denoted by $\nabla^2_{1,1} f$.
- If f_{xx} and f_{yy} are $[ii-p]$-gH differentiable, the Laplace operator of $f(x,y)$ is denoted by $\nabla^2_{2,2} f$.
- If f_{xx} is $[i-p]$-gH differentiable and f_{yy} is $[ii-p]$-gH differentiable, the Laplace operator of $f(x,y)$ is denoted by $\nabla^2_{1,2} f$.
- If f_{xx} is $[ii-p]$-gH differentiable and f_{yy} is $[i-p]$-gH differentiable, the Laplace operator of $f(x,y)$ is denoted by $\nabla^2_{2,1} f$.

Example 3.1. Consider the fuzzy-valued function $f(x,y) = \lambda y^2 \sin x$, where λ is a fuzzy number and $f(x,y) : [0, \frac{\pi}{2}] \times \mathbb{R}^+ \to \mathbb{R}_{\mathcal{F}}$; thus $f_{x_{gH}} = \lambda y^2 \cos x$ is $[i-p]$-gH differentiable and $f_{xx_{gH}} = -\lambda y^2 \sin x$ is $[ii-p]$-gH differentiable. Also $f_{y_{gH}} = 2\lambda y \sin x$ is $[i-p]$-gH differentiable and $f_{yy_{gH}} = 2\lambda \sin x$ is $[i-p]$-gH differentiable. Therefore grad $f(x,y)$ is denoted by $\nabla_{1,1} f$ and the Laplace operator of $f(x,y)$ is denoted by $\nabla^2_{2,1} f$.

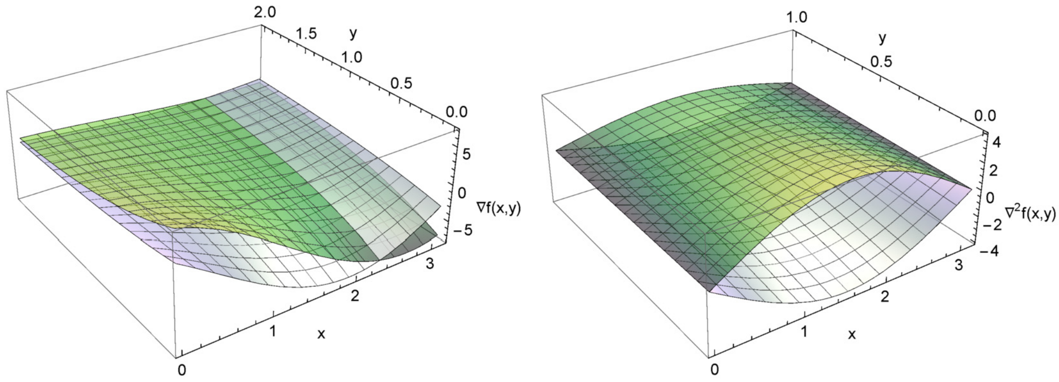

Fig. 1. Graph of Example 3.2: gradient of $f(x, y)$ for $\alpha = \frac{1}{2}$ (left) and Laplace operator of $f(x, y)$ for $\alpha = \frac{1}{2}$ (right).

Example 3.2. Consider the fuzzy-valued function $f(x, y) = \lambda e^{-y} \sin x$, where λ is a fuzzy number and $f(x, y) : [0, \pi] \times \mathbb{R}^+ \to \mathbb{R}_{\mathcal{F}}$; therefore $f_{x_{gH}} = \lambda e^{-y} \cos x$ is $[i - p]$-gH differentiable in $0 \le x < \frac{\pi}{2}$ and $[ii - p]$-gH differentiable in $\frac{\pi}{2} < x \le \pi$, and $f_{xx_{gH}} = -\lambda e^{-y} \sin x$ is $[ii - p]$-gH differentiable in $0 \le x \le \pi$. Thus $x = \frac{\pi}{2}$ is the type I switching point for $\partial_{x_{gH}} f(x, y)$ and we do not have a switching point for $\partial_{xx_{gH}} f(x, y)$ in $0 \le x \le \pi$, $y \in \mathbb{R}^+$.

On the other hand, $f_{y_{gH}} = -\lambda e^{-y} \sin x$ is $[ii - p]$-gH differentiable in $0 \le x \le \pi$, and $f_{yy_{gH}} = \lambda e^{-y} \sin x$ is $[i - p]$-gH differentiable in $0 \le x \le \pi$. Namely, we do not have a switching point for $\partial_{y_{gH}} f(x, y)$ and $\partial_{yy_{gH}} f(x, y)$ in $0 \le x \le \pi$, $y \in \mathbb{R}^+$.

Hence in view of the statements expressed, in $0 \le x \le \frac{\pi}{2}$, grad $f(x, y)$ is denoted by $\nabla_{1,2} f$ and the Laplace operator of $f(x, y)$ is denoted by $\nabla^2_{2,1} f$, and in $\frac{\pi}{2} \le x \le \pi$, grad $f(x, y)$ is denoted by $\nabla_{2,2} f$ and the Laplace operator of $f(x, y)$ is denoted by $\nabla^2_{2,1} f$ (see Fig. 1).

4. Fuzzy Green's identity

In this section we propose the fuzzy line integral and its fundamental fuzzy theorem. Then we prove the fuzzy Green's theorem, the fuzzy divergence theorem, and the fuzzy Green's identity. Moreover, some examples are given to show the applications and the efficiency of the proposed concepts.

Definition 4.1. If $f(x, y)$ is a fuzzy-valued function defined on smooth curve $\mathcal{C}$ given by the parametric curve equation $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$, where $r'(t) \ne 0$ is continuous and $x(t)$ and $y(t)$ are real-valued functions, then the fuzzy Riemann line integral of $f(x, y)$ along $\mathcal{C}$ is $\mathbb{I}_c = \int_C f(x, y)\, ds$ if for any $\varepsilon > 0$ there exists $\delta > 0$ such that for any division $S = \{[s_{i-1}, s_i],\ \zeta = (x_i^*, y_i^*)\}$ on $\mathcal{C}$ and $\triangle s_i = s_i - s_{i-1}$ with the norms of subarcs $\triangle s_i < \delta$ we have

$$D\left(\sum_s^{*} (s_i - s_{i-1}) \odot f(\zeta)\ ,\ \mathbb{I}_c\right) < \varepsilon, \tag{11}$$

where $\sum\limits_s^{*}$ denotes the fuzzy summation.

Note 7. We divide $\mathcal{C}$ into n subarcs $\triangle s_i$, so $[a, b]$ is divided into n subintervals $[t_{i-1}, t_i]$, where $t_i^* \in [t_{i-1}, t_i]$. Then we choose corresponding points $x_i^* = x(t_i^*)$ and $y_i^* = y(t_i^*)$ in the ith subarcs.

If $f(x, y)$ is a fuzzy continuous function, then (11) exists, and we can calculate the fuzzy line integral with respect to arc length as

$$\int_C f(x,y)\,ds = \int_a^b f(x(t),y(t)) \odot \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\; dt, \tag{12}$$

where if $s(t)$ is the length of $\mathcal{C}$ between $r(a)$ and $r(t)$, then $ds = |r'(t)| = \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$.

Example 4.1. Consider $\int_C (u \oplus v \odot x^2 y) ds$, where u and v are fuzzy numbers defined in terms of α-level by $u_\alpha = [-1+3\alpha, 5-2\alpha]$ and $v_\alpha = [1+2\alpha, 4-\alpha]$, and $\mathcal{C}$ is the upper half of the unit circle $x^2+y^2=1$.

The parametric equations of $\mathcal{C}$ are $x = \cos t,\ y = \sin t,\ 0 \le t \le \pi$. Therefore we have

$$\begin{aligned}\int_C (u \oplus v \odot x^2 y)\, ds &= \int_0^\pi (u \oplus v \odot \cos^2 t \sin t) \odot \sqrt{\sin^2 t + \cos^2 t}\, dt \\ &= \int_0^\pi u\, dt \oplus \int_0^\pi (v \odot \cos^2 t \sin t)\, dt \\ &= u \odot t\Big]_0^\pi \oplus v \odot \left(\frac{-\cos^3 t}{3}\right)\Big]_0^\pi .\end{aligned}$$

Hence the answer by α-levels is $[-1+3\alpha, 5-2\alpha]\pi + \frac{2}{3}[1+2\alpha, 4-\alpha]$.

Remark 5. We can express the fuzzy line integral with respect to x and y by the following formulas:

$$\int_C f(x,y)\,dx = \int_a^b f(x(t),y(t)) \odot x'(t)\ dt, \tag{13}$$

$$\int_C f(x,y)\,dy = \int_a^b f(x(t),y(t)) \odot y'(t)\ dt. \tag{14}$$

Definition 4.2. Suppose that $F(x,y) = P(x,y)\mathbf{i} \oplus Q(x,y)\mathbf{j}$ is a fuzzy-valued vector function defined by Definition 3.1. If the curve $\mathcal{C}$ is given by the vector parametric equation $r(t) = x(t)\mathbf{i} + y(t)\mathbf{j}$, $a \le t \le b$, where $r'(t) \ne 0$ and $F(x,y)$ is continuous on $\mathcal{C}$, we have

$$\int_C F(x,y)\,dr = \int_C F(x,y) \circledcirc T(t)\,ds = \int_a^b F(r(t)) \circledcirc r'(t)\ dt, \tag{15}$$

where $T(t) = \frac{r'(t)}{|r(t)|}$ is the unit tangent vector.

Example 4.2. To find the work done by the force field $F(x,y) = u \odot -x^2\ \mathbf{i} \oplus v \odot y\ \mathbf{j}$ (where u and v are fuzzy numbers defined in terms of α-level by $u_\alpha = [0.2+0.6\alpha, 1.5-0.7\alpha]$ and $v_\alpha = [0.5+1.5\alpha, 3-\alpha]$) on moving a particle along the upper half circle $r(t) = \cos t\ \mathbf{i} + \sin t\ \mathbf{j}$, $0 \le t \le \pi$, we must evaluate $\int_C F(x,y)\,dr$.

Since $x = \cos t,\ \ y = \sin t$ we get $F((r(t)) = u \odot -\cos^2 t\ \mathbf{i} \oplus v \odot \sin t\ \mathbf{j}$ and $r'(t) = -\sin t\ \mathbf{i} + \cos t\ \mathbf{j}$. In view of Definition 3.3 we have $F(r(t)) \circledcirc r'(t) = (u \odot \cos^2 t \sin t) \oplus (v \odot \cos t \sin t)$, where

$$\begin{cases} \cos^2 t \sin t \ge 0\,,\ \ \cos t \sin t \ge 0, & 0 \le t \le \frac{\pi}{2}, \\ \cos^2 t \sin t \ge 0\,,\ \ \cos t \sin t \le 0, & \frac{\pi}{2} \le t \le \pi. \end{cases}$$

Therefore

$$\int_C F(x,y)\,dr = \int_0^{\pi} F(r(t)) \circledcirc r'(t)\;dt$$
$$= \int_0^{\frac{\pi}{2}} \Big((u \odot \cos^2 t \sin t) \oplus (v \odot \cos t \sin t) \Big)\,dt \oplus \int_{\frac{\pi}{2}}^{\pi} \Big((u \odot \cos^2 t \sin t) \oplus (v \odot \cos t \sin t) \Big)\,dt$$
$$= (u \odot \frac{-\cos^3 t}{3}) \oplus (v \odot \frac{-\cos^2 t}{2})\Big]_0^{\frac{\pi}{2}} \oplus (u \odot \frac{-\cos^3 t}{3}) \oplus (v \odot \frac{-\cos^2 t}{2})\Big]_{\frac{\pi}{2}}^{\pi}$$
$$= (u \odot \frac{1}{3} \oplus v \odot \frac{1}{2}) \oplus (u \odot \frac{1}{3} \oplus v \odot \frac{-1}{2}).$$

Hence the answer by α-level is $\Big([0.2 + 0.6\alpha, 1.5 - 0.7\alpha] \cdot \frac{1}{3} + [0.5 + 1.5\alpha, 3 - \alpha] \cdot \frac{1}{2}\Big) + \Big([0.2 + 0.6\alpha, 1.5 - 0.7\alpha] \cdot \frac{1}{3} + [3 - \alpha, 0.5 + 1.5\alpha] \cdot \frac{-1}{2}\Big)$.

Remark 6. The parametric equation of $\mathcal{C}$ $(x = x(t),\;\; y = y(t), a \le t \le b)$ denotes an orientation with a positive direction corresponding to increasing value of t. Thus $-\mathcal{C}$ denotes the same points as curve $\mathcal{C}$ but with opposite orientation. Therefore we have $\int_{-C} f(x,y)\,dx = \ominus \int_C f(x,y)\,dx$ and $\int_{-C} f(x,y)\,dy = \ominus \int_C f(x,y)\,dy$.

Remark 7. $\int_{-C} f(x,y)\,ds = \int_C f(x,y)\,ds$, because by Definition 4.1, when we reverse the orientation of $\mathcal{C}$, $\triangle s_i \ge 0$ for all i.

Lemma 4.1. *Suppose that $\mathcal{C}$ is a piecewise-smooth curve and $F(x,y)$ is a fuzzy-valued function defined as follows:*

$$F(x,y) = \begin{cases} f_1(x,y) & on\ \mathcal{C}_1, \\ f_2(x,y) & on\ \mathcal{C}_2, \\ \vdots & \vdots \\ f_n(x,y) & on\ \mathcal{C}_n, \end{cases}$$

where $f_i(x,y) \in \mathbb{R}_{\mathcal{F}}$, $1 \le i \le n$, and $\mathcal{C}_i$, $1 \le i \le n$, are smooth curves. Since curve $\mathcal{C}$ is a union of $\mathcal{C}_1, \mathcal{C}_2, \dots, \mathcal{C}_n$, we have

$$\int_C F(x,y)\,dr = \int_{C_1} f_1(x,y)\,dr \oplus \int_{C_2} f_2(x,y)\,dr \oplus \dots \oplus \int_{C_n} f_n(x,y)\,dr. \tag{16}$$

Proof. The proof is almost identical to the proof of Lemma 2.3. □

Theorem 4.1. *(The fundamental theorem for the fuzzy line integral) Let $f(x,y)$ be a differentiable fuzzy-valued function on $\mathbb{R}^2$ and $\mathcal{C}$ be a smooth curve defined in Definition 4.1. Suppose that the gradient of $f(x,y)$ is continuous on $\mathcal{C}$ for any type ($\nabla_{i,j} f$, $\{i,j\} \in \{1,2\}$ exist). Then*

$$\int_C \nabla_{i,j} f(x,y)\,dr = f(r(b)) \ominus_{gH} f(r(a)) \qquad \{i,j\} \in \{1,2\}. \tag{17}$$

Proof. In view of Definition 4.2 we get

$$\int_C \nabla_{i,j} f(x,y)\,dr = \int_a^b \nabla_{i,j} f(r(t)) \circledcirc r'(t)\,dt.$$

First we suppose that $\nabla_{i,j} f := \nabla_{2,1} f$. According to Definition 3.3 we obtain

$$\int_C \nabla_{2,1} f(x,y)\, dr = \int_a^b \left(f_{x_{ii-gH}} \odot x'(t)\right) \oplus \left(f_{y_{i-gH}} \odot y'(t)\right)\, dt.$$

Consequently by Theorem 2.4 we obtain the following conditions:

1. If $x'(t), y'(t) > 0$: $\int_a^b \left(f_{x_{ii-gH}} \odot x'(t)\right) \oplus \left(f_{y_{i-gH}} \odot y'(t)\right)\, dt$.
2. If $x'(t), y'(t) < 0$: $\int_a^b \left(f_{x_{i-gH}} \odot x'(t)\right) \oplus \left(f_{y_{ii-gH}} \odot y'(t)\right)\, dt$.
3. If $x'(t) > 0,\ y'(t) < 0$: $\int_a^b \left(f_{x_{ii-gH}} \odot x'(t)\right) \oplus \left(f_{y_{ii-gH}} \odot y'(t)\right)\, dt$.
4. If $x'(t) < 0,\ y'(t) > 0$: $\int_a^b \left(f_{x_{i-gH}} \odot x'(t)\right) \oplus \left(f_{y_{i-gH}} \odot y'(t)\right)\, dt$.

It is evident from Theorem 2.6 and Lemma 2.1 that

$$\int_C \nabla_{2,1} f(x,y)\, dr = \int_a^b \frac{d}{dt} f(r(t))\, dt = f(r(b)) \ominus_{gH} f(r(a)).$$

The proof of another type of gradient is similar to this one. So equation (17) exists. □

Example 4.3. Suppose that $f(x,y) = u \odot x \oplus v \odot yx^2$, where u and v are the fuzzy numbers defined in terms of α-level by $u_\alpha = [1+2\alpha, 6-3\alpha]$ and $v_\alpha = [-1+3\alpha, 5-3\alpha]$, and $r(t) = \sin t\ \mathbf{i} + \cos t\ \mathbf{j},\ 0 \le t \le \frac{\pi}{2}$. Then $\nabla_{1,1} f = u\ \mathbf{i} \oplus v \odot x^2\ \mathbf{j}$ and $f\left(r(t)\right) = (u \odot \sin t) \oplus (v \odot \sin^2 t \cos t)$. Because of the statement expressed and Theorem 4.1 we conclude that

$$\begin{aligned}\int_C \nabla_{1,1} f(x,y)\, dr &= f(r(\frac{\pi}{2})) \ominus_{gH} f(r(0)) \\ &= u \ominus_{gH} 0 = u,\end{aligned}$$

where 0 is a singleton and the answer by α-level is $[1+2\alpha, 6-3\alpha]$.

Theorem 4.2. *(Fuzzy Green's theorem) Let $\mathcal{C}$ be a positively oriented, piecewise-smooth, simple closed curve and $\mathcal{D}$ be the region bounded by $\mathcal{C}$. Suppose that $P(x,y)$ and $Q(x,y)$ are fuzzy-valued functions on $\mathbb{R}^2$ whose partial derivatives are continuous on an open region that contains $\mathcal{D}$. Then*

$$\oint_C P(x,y)\, dx \oplus Q(x,y)\, dy = \iint_D Q_x(x,y) \ominus_{gH} P_y(x,y)\, dA. \tag{18}$$

Proof. If we prove the following equations, we can obtain (18):

$$\oint_C P(x,y)\, dx = \ominus_{gH} \iint_D P_y(x,y)\, dA \tag{19}$$

and

$$\oint_C Q(x,y)\, dy = \iint_D Q_x(x,y)\, dA. \tag{20}$$

To prove (19) we express $\mathcal{D}$ as a type region $\mathcal{D} = \{(x,y) | \ a \le x \le b,\ g_1(x) \le y \le g_2(x)\}$, where $a, b \in \mathbb{R}^2$ and $g_1(x),\ g_2(x)$ are real-valued continuous functions. Hence, by using Lemma 2.1, we have

$$\iint_D P_y(x,y)\,dA = \int_a^b \int_{g_1(x)}^{g_2(x)} P_y(x,y)\,dy\,dx$$
$$= \int_a^b P(x,g_2(x)) \ominus_{gH} P(x,g_1(x))\,dx. \quad (21)$$

Now we evaluate the left side of (19) by breaking up $\mathcal{C}$ as the union of four curves: $C_1,\ C_2,\ C_3,\ C_4$. On C_1 we take x as the parameter and obtain the parametric equations as $x = x$ and $y = g_1(x)$, $a \le x \le b$. Thus $\int_{C_1} P(x,y)\,dx = \int_a^b P(x,g_1(x))\,dx$. Notice that C_3 goes from right to left, and we can write the parametric equations of $-C_3$ as $x = x$ and $y = g_2(X)$, $a \le x \le b$. Therefore according to Remark 6 and Lemma 2.2,

$$\int_{C_3} P(x,y)\,dx = \ominus \int_{-C_3} P(x,y)\,dx = \ominus \int_a^b P(x,g_2(x))\,dx.$$

On C_2 and C_4, x is constant, so $dx = 0$ and we have

$$\int_{C_3} P(x,y)\,dx = \int_{C_4} P(x,y)\,dx = 0.$$

Hence according to Lemma 4.1

$$\oint_C P(x,y)\,dx = \int_{C_1} P(x,y)\,dx \oplus \int_{C_2} P(x,y)\,dx \oplus \int_{C_3} P(x,y)\,dx \oplus \int_{C_4} P(x,y)\,dx$$
$$= \int_a^b P(x,g_1(x))\,dx \oplus \ominus \int_a^b P(x,g_2(x))\,dx.$$

Consider $0 \ominus_{gH} \oint_C P(x,y)\,dx = \ominus_{gH} \oint_C P(x,y)\,dx$. We get

$$\ominus_{gH} \oint_C P(x,y)\,dx = \int_a^b P(x,g_2(x))\,dx \ominus_{gH} \int_a^b P(x,g_1(x))\,dx.$$

Hence (19) is proved.

By defining $\mathcal{D}$ as the region $\mathcal{D} = \{(x,y)|\ f_1(y) \le x \le f_2(y),\ a \le y \le b\}$, where $a, b \in \mathbb{R}$ and $f_1(y),\ f_2(y) \in \mathbb{R}_{\mathcal{F}}$, we can prove (20) in the same way:

$$\iint_D Q_x(x,y)\,dA = \int_a^b \int_{f_1(y)}^{f_2(y)} Q_x(x,y)\,dx\,dy$$
$$= \int_a^b Q(f_1(y),y) \ominus_{gH} Q(f_2(y),y)\,dy. \quad (22)$$

We break up $\mathcal{C}$ as the union of four curves: $C_1,\ C_2,\ C_3,\ C_4$. If we follow the same steps as in first part of the proof, $\int_{C_1} Q(x,y)\,dy = \int_{C_3} Q(x,y)\,dy = 0$ (y is constant on C_1 and C_2) and

$$\int_{C_2} Q(x,y)\,dy = \int_a^b Q(f_2(y),y)\,dy,$$

$$\int_{C_1} Q(x,y)\,dy = \ominus_{gH} \int_a^b Q(f_1(y),y)\,dy.$$

So we can evaluate the left side of (20) as follows:

$$\oint_C Q(x,y)\,dy = \int_{C_1} Q(x,y)\,dy \oplus \int_{C_2} Q(x,y)\,dy \oplus \int_{C_3} Q(x,y)\,dy \oplus \int_{C_4} Q(x,y)\,dy$$
$$= \int_a^b Q(f_2(y),y)\,dy \oplus \ominus \int_a^b Q(f_1(y),y)\,dy$$
$$= \int_a^b Q(f_2(y),y) \ominus_{gH} Q(f_1(y),y)\,dy.$$

Then equations (19) and (20) exist. □

Example 4.4. Consider $\oint_C (u \odot x^4)\,dx\ \oplus (v \odot xy)\,dy$, where u and v are fuzzy numbers defined in terms of α-level by $u_\alpha = [-1+3\alpha, 5-\alpha]$ and $v_\alpha = [2+\alpha, 7-2\alpha]$, and $\mathcal{C}$ is the triangular curve consisting of the line segments from $(0,0)$ to $(1,0)$, from $(1,0)$ to $(0,1)$, and from $(0,1)$ to $(0,0)$. To evaluate this fuzzy line integral we can use the fuzzy Green's theorem (Theorem 4.2). So we have

$$\oint_C (u \odot x^4)\,dx\ \oplus (v \odot xy)\,dy = \int_0^1 \int_0^{1-x} (v \odot y) \ominus_{gH} (u \odot 0)\,dydx$$
$$= \int_0^1 \int_0^{1-x} (v \odot y)\,dydx$$
$$= \int_0^1 \left(v \odot \frac{y^2}{2}\right)\Big]_0^{1-x}\,dx$$
$$= \int_0^1 \left(v \odot \frac{(1-x)^2}{2}\right)\,dx = \frac{1}{6} \odot v.$$

The answer by α-levels is $\frac{1}{6} \cdot [2+\alpha, 7-2\alpha]$.

Lemma 4.2. *According to the fuzzy Green's theorem we can conclude that*

$$\oint_C P(x,y)\,dy \ominus_{gH} Q(x,y)\,dx = \iint_D P_x(x,y) \oplus Q_y(x,y)\,dA. \tag{23}$$

Proof. It is enough to change the roles as follows: $P(x,y) := \ominus_{gH} Q(x,y)$ and $Q(x,y) := P(x,y)$. Then similarly to the fuzzy Green's theorem proof, we obtain

$$\oint_C \ominus_{gH} Q(x,y)\,dx = \iint_D Q_y(x,y)\,dA \text{ and } \oint_C P(x,y)\,dy = \iint_D P_x(x,y)\,dA.$$

By these equations we have

$$\oint_C P(x,y)\,dy \oplus \ominus_{gH} \oint_C Q(x,y)\,dx = \iint_D P_x(x,y)\,dA \oplus \iint_D Q_y(x,y)\,dA,$$

$$\oint_C P(x,y)\,dy \ominus_{gH} \oint_C Q(x,y)\,dx = \iint_D P_x(x,y) \oplus Q_y(x,y)\,dA,$$

and by Lemma 2.4 $\oint_C P(x,y)\,dy \ominus_{gH} Q(x,y)\,dx = \iint_D P_x(x,y) \oplus Q_y(x,y)\,dA$ is obtained. □

Theorem 4.3. *(Fuzzy divergence theorem) Let $\mathcal{C}$ be a smooth curve given by $r(t) = x(t)\boldsymbol{i} + y(t)\boldsymbol{j}$, $a \le t \le b$, and $F(x,y) = P(x,y)\boldsymbol{i} \oplus Q(x,y)\boldsymbol{j}$ be a differentiable fuzzy-valued vector function. It is sufficient that one of the components in $F(x,y)$ be a fuzzy function (this means one of P and Q has fuzziness and other one is a singleton). Then*

$$\oint_C F(x,y) \circledcirc n(t)\,ds = \iint_D \operatorname{div} F(x,y)\,dA, \tag{24}$$

where $\mathcal{D}$ is a region enclosed by $\mathcal{C}$ and $n(t) = \frac{y'(t)}{|r'(t)|}\boldsymbol{i} - \frac{x'(t)}{|r'(t)|}\boldsymbol{j}$ is an outward unit normal vector to $\mathcal{C}$.

Proof. As result of Definitions 3.3 and 4.2, equation (5), and Lemmas 2.5 and 4.2 we have

$$\begin{aligned}
\oint_C F(x,y) \circledcirc n(t)\,ds &= \int_a^b (F(x,y) \circledcirc n(t)) \circledcirc |r'(t)|dt \\
&= \int_a^b \left(P(x(t),y(t)) \odot \frac{y'(t)}{|r'(t)|} \oplus (-1)Q(x(t),y(t)) \odot \frac{x'(t)}{|r'(t)|} \right) \circledcirc |r'(t)|dt \\
&= \int_a^b \left(P(x(t),y(t)) \odot \frac{y'(t)}{|r'(t)|} \ominus_{gH} Q(x(t),y(t)) \odot \frac{x'(t)}{|r'(t)|} \right) \circledcirc |r'(t)|dt \\
&= \oint_C P(x,y)\,dy \ominus_{gH} Q(x,y)dx = \iint_D P_{x[i,ii-gH]} \oplus Q_{y[i,ii-gH]}\,dA \\
&= \iint_D \nabla_{i,j} \circledcirc F(x,y)\,dx,
\end{aligned}$$

where $\{i,j\} \in \{1,2\}$ and according to Definition 3.5 we know that $\nabla_{i,j} \circledcirc F(x,y) = \operatorname{div} F(x,y)$. In the case of a singleton, both types of differentiabilities are the same. Hence equation (24) exists. □

Theorem 4.4. *(Fuzzy Green's identity) Let $U(x,y)$ (for short U) be a fuzzy-valued function and $G(x,y)$ (for short G) be a real-valued function, the derivatives of which in any direction are uniform and continuous, and let $n(t)$ (for short n) be the outward unit normal vector to $\mathcal{C}$. We define surface $\mathcal{D}$ bounded by closed curve $\mathcal{C}$ by assumption of the fuzzy Green's theorem (Theorem 4.2). Then*

$$\iint_D (U \odot \nabla^2 G) \ominus_{gH} (G \odot \nabla^2_{i,j} U)\,dA = \oint_C \big((U \odot \nabla G) \ominus_{gH} (G \odot \nabla_{i,j} U)\big) \circledcirc n\,ds, \tag{25}$$

where $\{i,j\} \in \{1,2\}$.

Proof. By the divergence theorem we have $\oint_C F(x,y) \circledcirc n(t)\,ds = \iint_D \nabla_{i,j} \circledcirc F(x,y)\,dA$. We put $F := U \odot \nabla G$, where F is a fuzzy-valued vector function where one of the components is a fuzzy-valued function and $\frac{\partial G}{\partial x}$ and $\frac{\partial G}{\partial y}$ have the same sign. By Lemma 3.1 we get $\nabla \circledcirc F = \nabla \circledcirc (U \odot \nabla G) = U \odot (\nabla \circledcirc \nabla G) \oplus (\nabla_{i,j} U) \odot (\nabla G) = (U \odot \nabla^2 G) \oplus (\nabla_{i,j} U) \odot (\nabla G)$, for $\{i,j\} \in \{1,2\}$. Also $F \circledcirc n = (U \odot \nabla G) \circledcirc n$. So by using Theorem 4.3, we get

$$\iint_D (U \odot \nabla^2 G) \oplus \big((\nabla_{i,j} U) \odot (\nabla G)\big)\;dA = \oint_C (U \odot \nabla G) \circledcirc n\,ds. \tag{26}$$

Now define $F := G \odot \nabla_{i,j}U$, for $\{i, j\} \in \{1, 2\}$. So by Lemma 3.1 we have $\nabla \circledcirc F = \nabla \circledcirc (G \odot \nabla_{i,j}U) = (G \odot \nabla^2_{i,j}U) \oplus (\nabla G) \odot (\nabla_{i,j}U)$. Hence

$$\iint_D (G \odot \nabla^2_{i,j}U) \oplus (\nabla G) \odot (\nabla_{i,j}U)\, dA = \oint_C (G \odot \nabla_{i,j}U) \circledcirc n\, ds. \tag{27}$$

Since $(\nabla G) \odot (\nabla_{i,j}U) \ominus_{gH} \big((\nabla_{i,j}U) \odot (\nabla G)\big) = 0$, by subtracting equation (27) from equation (26), we get

$$\iint_D (U \odot \nabla^2 G) \ominus_{gH} (G \odot \nabla^2_{i,j}U)\, dA = \oint_C \big((U \odot \nabla G) \ominus_{gH} (G \odot \nabla_{i,j}U)\big) \circledcirc n\, ds. \quad \square$$

5. The fuzzy Poisson equation

In this section we investigate the fuzzy Poisson equation with Dirichlet boundary conditions of the following form:

$$\begin{aligned} &\nabla^2_{i,j}U = K(x, y) && \text{in } \mathcal{D}. \\ &U = \sigma(x, y) && \text{on } \partial\mathcal{D}, \end{aligned}$$

where $\{i, j\} \in \{1, 2\}$, $U = U(x, y)$, $K = K(x, y)$, $\sigma = \sigma(x, y) \in \mathbb{R}_{\mathcal{F}}$, and $\mathcal{D}$ is bounded. To achieve this, first we will prove the fuzzy maximum principle, then we will prove the uniqueness and stability theorem. Finally, the fundamental solution of the fuzzy Poisson equation is presented by use of a Green's function [18] and the fuzzy Green's identity theorem (Theorem 4.4).

5.1. Uniqueness and stability of the solution of the Dirichlet boundary conditions for the fuzzy Poisson equation

Theorem 5.1. *(Fuzzy maximum principle) Suppose that U satisfies $\nabla^2_{i,j}U = K$ in surface $\mathcal{D}$, where $U = U(x, y)$, $K = K(x, y) \in \mathbb{R}_{\mathcal{F}}$. Then U attains its maximum on its boundary $\partial\mathcal{D}$. Then* $\max_{\partial\mathcal{D}} U(x, y) = \max_{\overline{\mathcal{D}}} U(x, y)$.

Proof. We prove the theorem in two parts:

1. Suppose that $K(x, y) \succ 0$ in $\mathcal{D}$. Since U is continuous in $\overline{\mathcal{D}}$ $(\mathcal{D} \cup \partial\mathcal{D})$, U assumes its maximum there. Suppose for contradiction that U attains its maximum at point (x_0, y_0) in $\mathcal{D}$. Then, by Theorem 2.5, we have

$$\begin{aligned} &U_{x_{gH}}(x_0, y_0) = 0, && U_{y_{gH}}(x_0, y_0) = 0, \\ &U_{xx_{gH}}(x_0, y_0) \preceq 0, && U_{yy_{gH}}(x_0, y_0) \preceq 0. \end{aligned}$$

   This means $\nabla^2_{i,j}U \preceq 0$ at (x_0, y_0), which contradicts $K \succ 0$ in $\mathcal{D}$. Hence in this case U must attain its maximum on $\partial\mathcal{D}$. So

$$\max_{\partial\mathcal{D}} U(x, y) = \max_{\overline{\mathcal{D}}} U(x, y).$$

2. Suppose that $K(x, y) \succeq 0$. Let $M := \max_{\partial\mathcal{D}} U(x, y)$ and $\varepsilon > 0$. We define an auxiliary function $v(x, y) = U(x, y) \oplus \varepsilon(x^2 + y^2)$, for any $\varepsilon > 0$. Then $\nabla^2_{i,j}v = K \oplus 4\varepsilon \succ 0$ in $\mathcal{D}$. From the first part v attains its maximum on $\partial\mathcal{D}$; thus we have $v \preceq M \oplus \varepsilon R^2$ in $\mathcal{D}$ (where $U \preceq M$ on $\partial\mathcal{D}$ and R is the radius of the circle containing $\mathcal{D}$). This gives that $U \preceq v \preceq M \oplus \varepsilon R^2$.
   Since $\varepsilon > 0$ is arbitrary, let $\varepsilon \to 0$ to obtain $U \preceq M$ in $\mathcal{D}$ (i.e., if U satisfies $\nabla^2_{i,j}U = K \succeq 0$ in $\mathcal{D}$, then U cannot exceed M), thus the maximum value of U on $\partial\mathcal{D}$. $\square$

Remark 8. The same can be applied to the minimum principle.

Theorem 5.2. *Consider the Dirichlet boundary value problem*

$$\begin{aligned} &\nabla^2_{i,j}U = K(x, y) && \text{in } \mathcal{D}, \\ &U = \sigma(x, y) && \text{on } \partial\mathcal{D}, \end{aligned} \tag{28}$$

where $\{i, j\} \in \{1, 2\}$, $U = U(x, y)$, $K = K(x, y)$, $\sigma = \sigma(x, y) \in \mathbb{R}_{\mathcal{F}}$, *and* $\mathcal{D}$ *is bounded. This has at most one solution, and its solution is stable.*

Proof. Suppose that U_1, U_2 are two solutions of the Poisson equation (28). Let $V = U_1 \ominus_{gH} U_2$. Then V satisfies $\nabla^2 V = 0$ in $\mathcal{D}$ ($\nabla^2 U_1 \ominus_{gH} \nabla^2 U_2 = K \ominus_{gH} K = 0$) with $V = 0$ in $\partial\mathcal{D}$ ($U_1 = \sigma, U_2 = \sigma$). Then, by use of the maximum principle, $U_1 \ominus_{gH} U_2 = V \equiv 0$ in $\mathcal{D}$, and therefore $U_1 = U_2$.

For proof of stability of the solution, let U_1, U_2 satisfy

$$\begin{aligned} &\nabla^2_{i,j} U_{\{1,2\}} = K(x, y) \quad \text{in } \mathcal{D}, \\ &U_{\{1,2\}} = \sigma_{\{1,2\}}(x, y) \quad \text{on } \partial\mathcal{D}, \end{aligned}$$

where $\{i, j\} \in \{1, 2\}$, and let $D(\sigma_1 \ominus_{gH} \sigma_2,\ 0) = \varepsilon$ on $\partial\mathcal{D}$. Then $D(U_1 \ominus_{gH} U_2,\ 0) = \varepsilon$ on $\partial\mathcal{D}$. As before $V = U_1 \ominus_{gH} U_2$ must have its maximum and minimum values on $\partial\mathcal{D}$; hence $D(\sigma_1 \ominus_{gH} \sigma_2,\ 0) \le \varepsilon$ in $\overline{\mathcal{D}}$. So the solution is stable (i.e., small changes in boundary data lead to small changes in the solution). □

5.2. The fundamental solution for the fuzzy Poisson equation

Consider the fuzzy Poisson equation with Dirichlet boundary conditions in the general form, given by equation (28). We will solve problem (28) by using the fuzzy Green's identity and the Green's function.

Suppose that $\mathcal{L}(\mathbf{X})$ is a Laplace operator defined in Definition 3.6 ($\nabla^2 := \mathcal{L}$) and $U(\mathbf{X})$ is an unknown fuzzy-valued function and $K(\mathbf{X})$ is a known fuzzy-valued function, where $\mathbf{X} = (x, y)$. We can now represent problem (28) as follows:

$$\mathcal{L}(\mathbf{X})U(\mathbf{X}) = K(\mathbf{X}). \tag{29}$$

The solution to equation (29) can be written formally as

$$U(\mathbf{x}) = \mathcal{L}^{-1}K(\mathbf{X}),$$

where $\mathcal{L}^{-1}$, the inverse of $\mathcal{L}$, is an integral operator. $\mathcal{L}^{-1}$ is defined by use of the Green's function:

$$U(\mathbf{x}) = \mathcal{L}^{-1}K(\mathbf{X}) = -\iint\limits_{D} G(\mathbf{x}; \xi)K(\xi)\, d\xi, \tag{30}$$

where $G(\mathbf{x}; \xi)$ is the Green's function associated with $\mathcal{L}$ and depends on both the position vector $\mathbf{X} = (x, y)$ and a fixed location vector $\xi = (\xi_1, \xi_2)$.

We know the Dirac δ-function has the following properties:

$$\iint\limits_{D} \delta(\mathbf{X})\, d\mathbf{X} = 1 \text{ and } \iint\limits_{D} \delta(\mathbf{X} - \xi)\, h(\xi)\, d\xi = h(\xi). \tag{31}$$

By applying $\mathcal{L}$ to equation (30), we get

$$\mathcal{L}U(\mathbf{x}) = K(\mathbf{X}) = -\iint\limits_{D} \mathcal{L}G(\mathbf{x}; \xi)K(\xi)\, d\xi.$$

Therefore the Green's function defined as the solution of the Dirichlet boundary value problem is given by

$$\begin{aligned} &\mathcal{L}G(\mathbf{x}; \xi) = -\delta(\mathbf{X} - \xi) \quad \text{in } \mathcal{D}, \\ &G = 0 \quad \text{on} \partial\mathcal{D}, \end{aligned} \tag{32}$$

where $G(\mathbf{x}; \xi)$ is a real-valued function [21].

According to the fuzzy Green's identity theorem (Theorem 4.4) and replacing G by the Green's function $G(\mathbf{x}; \xi)$ introduced in equation (32), we have

$$\iint\limits_{D} (U(\mathbf{X}) \odot \nabla^2 G(\mathbf{X}; \xi)) \ominus_{gH} (G(\mathbf{X}; \xi) \odot \nabla^2_{i,j} U(\mathbf{X}))\, dA \tag{33}$$

$$= \oint\limits_{\partial D} \big((U(\mathbf{X}) \odot \nabla G(\mathbf{X}; \xi)) \ominus_{gH} (G(\mathbf{X}; \xi) \odot \nabla_{i,j} U(\mathbf{X})\big) \circledcirc n\, ds.$$

According to the fuzzy Poisson equation (28) and the Green's function equation (32) we get

$$\iint_D (U(\mathbf{X}) \odot \ -\delta(\mathbf{X}-\xi)) \ominus_{gH} (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA \tag{34}$$

$$= \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds.$$

By Lemma 2.4 and Dirac δ-function properties (31) we have

$$(-1)\cdot U(\xi) \ominus_{gH} \iint_D (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA = \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds. \tag{35}$$

Hence

- if $\ominus_{gH}$ is case (i), we obtain

$$(-1)\cdot U(\xi) = \iint_D (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA \oplus \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds$$

  or

$$U(\xi) = (-1)\cdot \iint_D (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA \oplus (-1)\cdot \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds. \tag{36}$$

- if $\ominus_{gH}$ is case (ii), we obtain

$$(-1)\cdot U(\xi) = \iint_D (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA \ominus (-1)\cdot \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds$$

  or

$$U(\xi) = (-1)\cdot \iint_D (G(\mathbf{X};\xi) \odot K(\mathbf{X}))\, dA \ominus \oint_{\partial D} (\sigma(\mathbf{X}) \odot \nabla G(\mathbf{X};\xi)) \circledcirc n\, ds. \tag{37}$$

  In equations (36) and (37), if $U(\xi)$ is a fuzzy number, then the solution of the Poisson equation exists; otherwise a solution does not exist.

Example 5.1. (The fundamental solution for the fuzzy Poisson equation on a two-dimensional disc) Consider the fuzzy Poisson equation with Dirichlet boundary conditions as follows:

$$\begin{array}{ll} \nabla^2_{i,j} U = K(x,y) & \text{in } \mathcal{D}, \\ U = \sigma(x,y) & \text{on } \partial\mathcal{D}, \end{array} \tag{38}$$

where $\{i,j\} \in \{1,2\}$, $\mathcal{D} = \{(x,y) \mid x^2+y^2 \le a^2\}$, and $U(x,y),\ K(x,y),\ \sigma(x,y) \in \mathbb{R}_{\mathcal{F}}$. In the polar plane on a two-dimensional disc, the Green's function is evaluated [18] as

$$G(x,y;\xi_1,\xi_2) = G(r,\theta;r^*,\theta^*) = -\frac{1}{4\pi}\ln\left(\frac{a^2}{r^{*2}} \cdot \frac{r^2+{r^*}^2 - 2rr^*\cos(\theta-\theta^*)}{r^2+\frac{a^4}{r^{*2}} - 2r\frac{a^2}{r^*}\cos(\theta-\theta^*)}\right),$$

where $x = r\cos\theta,\ y = r\sin\theta,\ 0 \le \theta \le \pi$. $\mathcal{C}(\theta) = a\cos\theta\ i + a\sin\theta\ j$ is the vector equation of curve $\partial\mathcal{D}$, and we denote $G(r,\theta;r^*,\theta^*)$ by G (for short). Therefore

$$\nabla G \circledcirc n = \frac{-a}{2\pi}\left(\frac{1-(\frac{r^*}{a})^2}{r^{*2}+a^2-2ar^*\cos(\theta-\theta^*)}\right).$$

In view of the general form of the solution for the fuzzy Poisson equation given by equations (36) and (37) we have two types of solution as follows:

$$U(r^*,\theta^*) = \frac{-1}{4\pi}\int_0^{2\pi}\int_0^{a} \ln\left(\frac{a^2}{r^{*2}}\cdot\frac{r^2+r^{*2}-2rr^*\cos(\theta-\theta^*)}{r^2+\frac{a^4}{r^{*2}}-2r\frac{r^2}{r^*}\cos(\theta-\theta^*)}\right)\odot K(r,\theta)\, r dr d\theta$$
$$\oplus\frac{1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^{*2})}{a^2-r^{*2}-2ar^*\cos(\theta-\theta^*)}\odot\sigma(\theta)\, d\theta \tag{39}$$

or

$$U(r^*,\theta^*) = \frac{-1}{4\pi}\int_0^{2\pi}\int_0^{a} \ln\left(\frac{a^2}{r^{*2}}\cdot\frac{r^2+r^{*2}-2rr^*\cos(\theta-\theta^*)}{r^2+\frac{a^4}{r^{*2}}-2r\frac{a^2}{r^*}\cos(\theta-\theta^*)}\right)\odot K(r,\theta)\, r dr d\theta$$
$$\ominus\frac{-1}{2\pi}\int_0^{2\pi}\frac{(a^2-r^{*2})}{a^2-r^{*2}-2ar^*\cos(\theta-\theta^*)}\odot\sigma(\theta)\, d\theta. \tag{40}$$

6. Conclusion

In this study, the fuzzy Poisson equation with Dirichlet boundary conditions (boundary value problem) was investigated in detail. Its uniqueness and stability were proved by use of the fuzzy maximum principle. Then its fundamental fuzzy solution was provided. To achieve these results, some new concepts such as a fuzzy-valued vector function, fuzzy gradient, divergence, and Laplace operators, and a fuzzy line integral were studied. Then the fuzzy Green's theorem and the fuzzy divergence theorem and the fuzzy Green's identity were proved. Consequently, the fuzzy solution of the fuzzy Poisson equation was obtained by our applying an inverse operator. As a result, an integral equation was obtained; therefore the fundamental fuzzy solution was presented by use of the fuzzy Green's identity and a Green's function. The aforesaid fundamental solution can be calculated by numerical methods that were discussed in this study. Future research will be concerned with adopting these results to calculate the fuzzy Poisson solution numerically. In this article, all the results obtained are original results that have been extended to the fuzzy form in such a way that their restriction (one level) results in the real mode, and this is a criterion for establishing the authenticity of results.

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